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HOW TO CONTROL IF EVEN EXPERTS ARE NOT SURE: ROBUST FUZZY CONTROL

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ABSTRACT

In real life, the degrees of certainty that correspond to one of the same expert can differ drastically, and fuzzy control algoirthms translate these different degrees of uncertainty into different control strategies. In such situation, it is reasonable to choose a fuzzy control methodology that is the least vulnerable to this kind of uncertainty. We show that this "robustness" demand leads to min and max for &− and ∨−operations, to $1-x$ for negation, and to centroid as a defuzzification procedure.

INTRODUCTION

In case we do not have the precise knowledge of a controlled system, we are unable to apply traditional control theory. In such cases, we can find an expert who is good at control, extract as many rules as possible from him, and try to transform these rules into the precise control strategy. Zadeh and Mamdani initiated a methodology for such a translation ([4, 14]) that is based on *fuzzy set theory* [18] and is therefore called *fuzzy control* (see, e.g., the surveys [1, 12, 17]). In order to apply this methodology, we must:

1) describe the expert's uncertainty about every natural-language term A (such as "small") that he uses while describing the control rules; this is done by ascribing to every possible value x of the related physical quantity a value $\mu_A(x)$ from the interval [0,1] that describes to what extent this expert believes that x satisfies the property A (e.g., $\mu_{small}(0.3)$ is his degree of belief that 0.3 is small). The resulting function μ_A is called a *membership function*;
2) experts' rules contain natural-language words combined by logical connectives (e.g., "*if* x is small, *and* $\dot{x}$ is medium, *then* u must be small"). Therefore, we must be able to estimate the experts' degree of belief in $A\&B$, $A \vee B$, $\neg A$ (where $\neg$ stands for "not") from the known values of degrees of belief of A and B. In other words, we must describe the fuzzy analogues of &, ∨, and ¬ to combine the original membership functions into a membership function $\mu_C(u)$ for control;
3) finally, we must transform this membership functions into an actual control value by a proper *defuzzification* procedure.

As concerns the first stage, there exist several methods that allow us to ask several questions to an expert or experts and come out with the desired values of membership functions (see, e.g., [6, 8]). This makes perfect sense if the experts (whom we ask) give "yes" or "no" answers to all these questions, i.e., when they are absolutely sure of what they are doing. They may be unable to describe their control strategy in precise mathematical terms, but they are absolutely confident in what they are doing (a good example is a person driving his car: he has no doubts about his ability to drive, but he usually cannot formulate his strategy in precise terms).

However, if we are planning a trip to the unknown (e.g., a mission to Mars), then operators are often not that confident in their control abilities. For example, they can formulate a rule in terms of a certain angle being small, but they are uncertain of whether, say, 10^o is a small angle or not. As a result, the values of membership functions that we extract from the same expert can differ drastically. Different membership functions, in their turn, can lead to drastically different control strategies, with different quality of the resulting control.

This situation can be viewed as one step further away from the precision of traditional control:

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precise knowledge $\longrightarrow$ *uncertain knowledge with known degrees of certainty* $\longrightarrow$
uncertain knowledge with uncertain degrees of certainty

What to do in these maximally uncertain situations? Since fuzzy control proved to be a very efficient methodology [1, 12, 17], we still want to use, but we must now be very cautious in choosing $\&-$, $\vee-$, and $\neg-$ operations, and in choosing a defuzzification procedure. In all these choices, we want to result to be as *robust* to the possible changes in the values of membership functions as possible. In other words, we want to develop *robust control.*

ROBUSTNESS OF &– AND ∨–OPERATIONS

Let us first analyze the case of $\&-$ and $\vee-$operations (this section subsumes [15]).

The first paper by L. Zadeh [18] that introduced this approach to knowledge representation proposed $\min(a,b)$ and ab as $\&-$operations, and $\max(a,b)$ and $a+b-ab$ as $\vee-$operations. Zadeh himself stressed that these operations "are not the only operations in terms of which the union and intersection can be defined", and "which of these ... definitions are more appropriate depends on the context" [19, pp. 225-226]. Since then several dozens different $\&-$ and $\vee-$operations have been proposed and successfully used. Some operations have been discovered empirically while working on real expert systems (e.g., the famous MYCIN [3]) or while analyzing commonsense reasoning [16, 20]); some of them were proposed on a more theoretical basis (see, e.g., [6, 8]). A survey of such operations is given in [11].

Definition 1.
(i). By a *binary operation* (or *operation* for short) we mean a function $f(a,b)$ from $[0,1]\times[0,1]$ into $[0,1]$.
(ii). A binary operation is called a *$\&-$operation* if the following conditions are true:
$f(0,0)=f(0,1)=f(1,0)=0, f(1,1)=1$; $f(a,b)=f(b,a)$ for all a,b; $f(a,b)\le a$ for all a and b.
(iii). A binary operation is called an *$\vee-$operation* if the following conditions are true:
$f(0,0)=0, f(0,1)=f(1,0)=f(1,1)=1$; $f(a,b)=f(b,a)$ for all a,b; $f(a,b)\ge a$ for all a and b.

Remark. The above binary operations are slightly more general than the usual $t-$norms and $t-$conorms in the literature [7].

Definition 2. Suppose that a binary operation $f(a,b)$ is given. We say that a $\delta-$input uncertainty leads to a $\le\alpha-$output error, if for every a,a',b,b', for which $|a-a'|\le\delta$ and $|b-b'|\le\delta$, we have $|f(a,b)-f(a',b')|\le\alpha$.

Remark. In other words, if $a'\in[a-\delta,a+\delta]$, and $b'\in[b-\delta,b+\delta]$, then $f(a',b')\in[f(a,b)-\alpha,f(a,b)+\alpha]$. For example, if the interval of possible values for the expert's degree of belief $t(A)$ in some statement A is $[a-\delta,a+\delta]$, and for some other statement B the interval of possible values of $t(B)$ is $[b-\delta,b+\delta]$, then the interval of possible values of $t(A\&B)$ must be contained in $[f(a,b)-\alpha,f(a,b)+\alpha]$.

Definition 3. Suppose that $f(a,b)$ is a binary operation, and $\delta>0$ is a positive real number. By a *$\delta-$robustness* of an operation $f(a,b)$ we mean the smallest of real numbers α, for which a $\delta-$input uncertainty leads to a $\le\alpha-$output error. The $\delta-$robustness of an operation $f(a,b)$ will be denoted by $r_f(\delta)$.

Remark. It is easy to check that $r_f(\delta)=\sup\{|f(a,b)-f(a',b')|:|a-a'|\le\delta,|b-b'|\le\delta\}$. When f is continuous, $r_f(\delta)$ is the well-known modulus of continuity of f [13]. The above sup is in fact max: see Proposition 1 below; its proof, as well as all the proofs of the results are in the last Section for easy reading.

Proposition 1. *For every operation $f(a,b)$, and for every $\delta>0$, there exists a $\delta-$robustness (i.e., the smallest of real numbers α, for which a $\delta-$input uncertainty leads to a $\le\alpha-$output error).*

In order to compare different operations in terms of robustness, we will proceed as in standard decision theory, where $r_f(\delta)$ plays the role of the "risk" (see, e.g., [2]).

Definition 4. We say that operations $f(a,b)$ and $g(a,b)$ are *equally robust* if for every δ, $r_f(\delta)=r_g(\delta)$. We say that an operation $f(a,b)$ is *more robust* than an operation $g(a,b)$, if for every δ, $r_f(\delta)\le r_g(\delta)$, and at least for one $\delta>0$, $r_f(\delta)<r_g(\delta)$.

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Definition 5. We say that an &−operation $f(a,b)$ is *the most robust &−operation*, if it is either more robust, or equally robust than any other &−operation. We say that an $\vee$−operation $g(a,b)$ is *the most robust* $\vee$*−operation*, if it is either more robust, or equally robust than any other $\vee$−operation.

Remark. In other words, we say that an operation is the most robust, if the resulting intervals of uncertainty are the smallest possible.

Theorem 1. *$f(a,b) = \min(a,b)$ is the most robust &−operation.*

Theorem 2. *$f(a,b) = \max(a,b)$ is the most robust $\vee$−operation.*

Remarks.
1. In [9–11] general optimization problem are analyzed on the set of all possible &− and $\vee$− operations. As a result of this mathematical analysis, lists are given that include all &− and $\vee$− operations that can be optimal under reasonable optimality criteria. Our Theorems 1 and 2 are in good accordance with that general result, because both min and max are elements of those lists.
2. Similar questions of robustness in the context of neural networks are analyzed in [5].
3. It is interesting to know to what extent the robustness functions that correspond to min and max are smaller than those of the other binary operations: are they smaller in a few points only, or essentially smaller for all δ? The answer is given by the following Theorems:

Theorem 3. *Suppose that $f(a,b)$ is an &−operation, and $f(a,b)$ is different from* min. *Then there exists a positive real number $\Delta > 0$ and positive real number $C < 1$ such that for all $\delta < \Delta$, $r_{\min}(\delta) \le C r_f(\delta)$.*

Theorem 4. *Suppose that $f(a,b)$ is an $\vee$−operation, and $f(a,b)$ is different from* max. *Then there exists a positive real number $\Delta > 0$ and positive real number $C < 1$ such that for all $\delta < \Delta$, $r_{\max}(\delta) \le C r_f(\delta)$.*

The following Theorem describes δ−robustness for several other operations:

Theorem 5.
1) if $f(a,b) = ab$, then $r_f(\delta) = 2\delta - \delta^2$;
2) if $f(a,b) = a + b - ab$, then $r_f(\delta) = 2\delta - \delta^2$;
3) if $f(a,b) = \min(a+b,1)$, then $r_f(\delta) = \min(2\delta,1)$.

Remarks.
1. Unlike Theorems 1 and 2, this result uses traditional techniques of interval mathematics.
2. In these three cases, $\lim_{\delta\to 0} r_f(\delta)/r_{\min}(\delta) = 2$, so in Theorems 3 and 4 we can take $C = 1/2 + \alpha$ for arbitrary small $\alpha > 0$.
3. From Theorem 5 we can conclude that the operations ab and $a + b - ab$ are equally robust, and that $f(a.b) = \min(a+b,1)$ is more robust than both of them.
4. The fact that ab and $a + b - ab$ are equally robust stems from the fact that in general *dual* operations have the same modulus of continuity, where $g(a,b)$ is *dual* to an $f(a,b)$ if $g(a,b) = 1 - f(1-a, 1-b)$.

Proposition 2. *Dual operations are equally robust.*

ROBUSTNESS OF NEGATION OPERATIONS

Definition 6. By a *negation operation* we mean a decreasing function $f : [0,1] \to [0,1]$ such that $f(0) = 1$, $f(1) = 0$ and $f(f(x)) = x$.

Definition 7. Suppose that a negation operation $f(a)$ is given. We say that a δ−input uncertainty leads to a $\le \alpha$−output error, if for every a, a', for which $|a - a'| \le \delta$, we have $|f(a) - f(a')| \le \alpha$. By a *$\delta$−robustness* of an operation $f(a)$ we mean the smallest of real numbers α, for which a δ−input uncertainty leads to a $\le \alpha$−output error. The δ−robustness of an operation $f(a)$ will be denoted by $r_f(\delta)$.

Theorem 6. *$f(x) = 1 - x$ is the most robust negation operation.*

Theorem 7. *Suppose that $f(x)$ is a negation operation, and $f(x)$ is different from $f_0(x) = 1 - x$. Then there exists a positive real number $\Delta > 0$ and positive real number $C < 1$ such that for all $\delta < \Delta$, $r_{f_0}(\delta) \le C r_f(\delta)$.*

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ROBUST CHOICE OF A DEFUZZIFICATION

Motivation. A *defuzzification* is a procedure D that transforms a membership function $\mu(x)$ into a real number $D(\mu)$. In the present paper, we will compare two defuzzification rules that are most wide spread in fuzzy control theory: centroid rule and center-of-maximum rule (see description below). Both defuzzification rules are applicable in case a membership function $\mu(x)$ has a compact support (i.e., is equal to 0 outside some interval), and is not everywhere equal to 0. Let us recall the definitions.

Definition 8. Let us fix an interval $[a, b]$, and consider only membership functions that are equal to 0 outside this interval, and that are not everywhere equal to 0. The set of all such membership functions will be denoted by M. A mapping D from M to $[a, b]$ is called a *defuzzification*. By a *centroid defuzzification* we mean a mapping that transforms $\mu(x)$ into a number $D_C(\mu) = (\int x\mu(x)\,dx)/(\int \mu(x)\,dx)$. By a *center-of-maximum* defuzzification we mean a mapping that transforms $\mu(x)$ from M into a number $D_{COM}(\mu) = 1/2(m_- + m_+)$, where $m_- = \inf\{x : \mu(x) = \max_y(\mu(y))\}$, and $m_+ = \sup\{x : \mu(x) = \max_y(\mu(y))\}$.

Motivation of the following definition. In this case, it turns out that it is not necessary to compare the numerical estimates of robustness, because it turns out that one of these procedures is simply robust, and another is not in the following sense. It is reasonable to say that a procedure is *robust* if small deviation of $\mu(x)$ lead to small deviations in the result. Here "small deviations" mean that $|\mu(x) - \mu'(x)| \leq \delta$ for some small number δ, or, in other words, that $\rho(\mu, \mu') \leq \delta$, where we denoted $\rho(\mu, \mu') = \sup_x |\mu(x) - \mu'(x)|$. So, in mathematical terms, robustness just means that a mapping D must be continuous in this metric ρ. Let us first remind the definition of continuity:

Definition 9. A mapping D from M to $[a, b]$ is called *continuous* at the point μ with respect to metric ρ, if for every ε there exists a δ such that if $\rho(\mu, \mu') \leq \delta$, then $|D(\mu) - D(\mu')| \leq \varepsilon$. If D is continuous in all the points of M, it is called a *continuous* mapping.

Definition 10. We say that a defuzzification is *robust*, if it is a continuous mapping from DM (with metric ρ) into the interval $[a, b]$.

Theorem 8. *Centroid defuzzification is robust, while center-of-mass defuzzification is not.*

ROBUST NORMALIZATION

Motivation. In some cases, before making a decision an auxiliary operation is performed with a membership function $\mu_C(u)$ that is called a *normalization*. The reason for this operation is that for many notion from natural language, there is a value about which all the experts (or at least the vast majority of them) agree that this value satisfies the desired property: for example, for "negligible" it is 0, for "big" it is 1000 (or 10^6 if 1000 is not enough). So, for the corresponding membership functions $\mu(x)$, there exists a value x_0 for which $\mu(x_0) = 1$, hence $\sup_x \mu(x) = 1$.

However, after applying the $\&-$, $\vee-$ and $\neg-$operations, we sometimes obtain a membership function $\mu(x)$, for which $\sup_x \mu(x) < 1$, and which is thus difficult to interpret. So, before we apply a defuzzification procedure to it, we first want to *normalize* this membership function, i.e., apply some transformation $t : [0, v] \to [0, 1]$ and get a new function $\mu'(x) = t(\mu(x))$ whose biggest value is already equal to 1. Usually, the function $t(x) = x/v$ is taken. The question is: which of the possible normalization procedures is the most robust?

Definition 11. Assume that a number $v < 1$ is given. By a *normalization* we mean an increasing function $f : [0, v] \to [0, 1]$ such that $f(0) = 0$ and $f(v) = 1$.

Definition 12. Suppose that a normalization $f(a)$ is given. We say that a $\delta-$input uncertainty leads to a $\leq \alpha-$output error, if for every a, a', for which $|a - a'| \leq \delta$, we have $|f(a) - f(a')| \leq \alpha$. By a *$\delta-$robustness* of a normalization $f(a)$ we mean the smallest of real numbers α, for which a $\delta-$input uncertainty leads to a $\leq \alpha-$output error. The $\delta-$robustness of a normalization $f(a)$ will be denoted by $r_f(\delta)$.

Remark. As the following result shows, no normalization is most robust than the others:

Proposition 3. *For every normalization f there exists another normalization g and a value $\delta > 0$ such that $r_g(\delta) < r_f(\delta)$.*

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Remark. This results is not as discouraging as it may seem at first glance. The reason is as follows. As one can see from the proof, this δ can be big. From the purely mathematical viewpoint, this is quite reasonable. However, the entire purpose of analyzing robustness of fuzzy operations is to take into consideration the uncertainty with which experts can estimate their own beliefs, and this uncertainty is usually small. So it makes sense to compare δ–robustness of different operations only for small δ. So we arrive at the following definition.

Definition 13. We say that a function $f(\delta)$ is *asymptotically smaller* than a function $g(\delta)$ if there exists a $\Delta > 0$ such that for $\delta < \Delta$, $f(\delta) < g(\delta)$.

Definition 14. We say that a normalization $f(x)$ is *asymptotically more robust* than a normalization $g(x)$ if the robustness $r_f(\delta)$ of $f(x)$ is asymptotically smaller than the robustness $r_g(\delta)$ of $g(x)$. We say that a normalization $f(x)$ is the *most asymptotically robust* normalization, if it is asymptotically more robust than any other normalization.

Remark. In doing this, we also proceed as in standard decision theory, with $r_f(\delta)$ playing the role of the "risk" (see, e.g., [BG79]).

Theorem 9. *$f(x) = x/v$ is the most asymptotically robust normalization.*

Theorem 10. *Suppose that $f(x)$ is a normalization, and $f(x)$ is different from $f_0(x) = x/v$. Then there exists a positive real number $\Delta > 0$ and positive real number $C < 1$ such that for all $\delta < \Delta$, $r_{f_0}(\delta) \le C r_f(\delta)$.*

ROBUST CHOICE OF MEMBERSHIP FUNCTIONS

Motivation. All the above considerations are about the case when the experts can be uncertain, but the inputs for the control decision (i.e., the values of x, $\dot{x}$, etc) are considered to be precise. In real-life situations, especially in the case of the future space missions, it is important to take into considerations that the input data can also be imprecise. In this case, we want to choose membership functions in such a way that the change in an input value x will lead to the smallest possible change in the value of $\mu(x)$ (and thus in the resulting control). In other words, we want to guarantee that the interval of possible values of $\mu(x)$ is the least possible.

We want to use this idea to choose the most robust extrapolation procedure for membership functions. In other words, when we have a fuzzy notion for which we want to describe a membership function, we describe when this notion is absolutely true, and when it is absolutely false (i.e., when the membership function is equal to 1 and 0) and get all other values of membership function by extrapolation.

In fuzzy control, four types of natural-languages terms (*fuzzy variables*) variables are mainly used:

1) Variables like "negligible", where one can name a value x_0 for which the corresponding property is absolutely true ($\mu(x_0) = 1$), (for negligible it is $x_0 = 0$), and the values x_- and x_+ such that for $x < x_-$ and $x > x_+$ the corresponding property is absolutely false (e.g., values with $x < x_-$ or $x > x_+$ are absolutely not negligible).
2) (similar case) Variables, for which we can name an interval $[a_-, a_+]$, inside which the corresponding property is absolutely true, and a bigger interval $[x_-, x_+]$, outside which this property is absolutely false (the first case can be considered as a particular case of this one, when $a_- = a_+ = x_0$).
3) Variable like "positive big", for which we can name values $x_- < x_+$ such that for $x < x_-$ the corresponding property is absolutely false, and for $x > x_+$ this property is absolutely true.
4) Variable like "negative big", for which we can name values $x_- < x_+$ such that for $x < x_-$ the corresponding property is absolutely true, and for $x > x_+$ this property is absolutely false.

In the first and second cases, usually the intervals are symmetric, i.e., in the first case, $x_+ - x_0 = x_0 - x_-$, and, in the second case, $x_+ - a_+ = a_- - x_-$.

Now we are ready for the definitions.

Definition 15. Suppose that $x_- < x_0 < x_+$ are real numbers such that $x_+ - x_0 = x_0 - x_-$. By an *extrapolated membership function of type 1* we mean a membership function $\mu(x)$ such that $\mu(x_0) = 1$, and $\mu(x) = 0$ if $x < x_-$ or $x > x_+$.

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Remark. The simplest possible extrapolation (that is often used in fuzzy control) is a piecewise-linear extrapolation, that in this case leads to a *triangular* function:

Definition 16. Assume that x_-, x_0 and x_+ are given. By a *triangular function* we mean a membership function $\mu_0(x)$ that is equal to $1 - |x - x_0|/(x_+ - x_0)$ for $x \in [x_-, x_+]$ and 0 outside this interval.

Definition 17. Suppose that a membership function $\mu(x)$ is given. We say that a $\delta-$input data uncertainty leads to a $\leq \alpha-$output error, if for every a, a', for which $|a - a'| \leq \delta$, we have $|f(a) - f(a')| \leq \alpha$. By a *$\delta-$robustness* of a membership function $\mu(x)$ we mean the smallest of real numbers α, for which a $\delta-$input data uncertainty leads to a $\leq \alpha-$output error. The $\delta-$robustness of a membership function $\mu(x)$ will be denoted by $r_\mu(\delta)$.

Theorem 11. *For any given x_-, x_0, x_+, triangular membership function $\mu_0(x)$ is asymptotically the most robust extrapolated membership function of type 1.*

Definition 18. Suppose that $x_- < a_- < a_+ < x_+$ are real numbers such that $x_+ - a_+ = a_- - x_-$. By an *extrapolated membership function of type 2* we mean a membership function $\mu(x)$ such that $\mu(x) = 1$ for $x \in [a_-, a_+]$, and $\mu(x) = 0$ if $x < x_-$ or $x > x_+$.

Definition 19. Assume that x_-, a_-, a_+ and x_+ are given. By a *trapezoidal function* we mean a membership function $\mu_0(x)$ that is equal to 1 for $x \in [a_-, a_+]$, to 0 outside the interval $[x_-, x_+]$, to $(x - x_-)/(a_- - x_-)$ for $x \in [x_-, a_-]$, and to $1 - (x - a_+)/(x_+ - a_+)$ for $x \in [a_+, x_+]$.

Theorem 12. *For any given x_-, a_-, a_+, x_+, trapezoidal membership function $\mu_0(x)$ is asymptotically the most robust extrapolated membership function of type 2.*

Definition 20. Suppose that $x_- < x_+$ are real numbers. By an *extrapolated membership function of type 3* we mean a membership function $\mu(x)$ such that $\mu(x) = 0$ for $x < x_-$, and $\mu(x) = 1$ if $x > x_+$.

Definition 21. Assume that x_- and x_+ are given. By a *piecewise-linear function (of type 3)* we mean a membership function $\mu_0(x)$ that is equal to 0 for $x < x_-$, to $(x - x_-)/(x_+ - x_-)$ for $x \in [x_-, x_+]$, and to 0 for $x > x_+$.

Theorem 13. *For any given x_- and x_+, piecewise-linear membership function $\mu_0(x)$ is asymptotically the most robust extrapolated membership function of type 3.*

Definition 22. Suppose that $x_- < x_+$ are real numbers. By an *extrapolated membership function of type 4* we mean a membership function $\mu(x)$ such that $\mu(x) = 1$ for $x < x_-$, and $\mu(x) = 0$ if $x > x_+$.

Definition 23. Assume that x_- and x_+ are given. By a *piecewise-linear function (of type 4)* we mean a membership function $\mu_0(x)$ that is equal to 1 for $x < x_-$, to $1 - (x - x_-)/(x_+ - x_-)$ for $x \in [x_-, x_+]$, and to 1 for $x > x_+$.

Theorem 14. *For any given x_- and x_+, piecewise-linear membership function $\mu_0(x)$ is asymptotically the most robust extrapolated membership function of type 4.*

PROOFS

Proof of Proposition 1. The set S of all real numbers α, for which a $\delta-$input uncertainty leads to a $\leq \alpha-$output error, is bounded from below (by 0), and therefore, has an infinum (the greatest lower bound) r. r is the value of $\delta-$robustness. Indeed, since r is the greatest lower bound of the set S, for every positive integer k there exists a number $r_k \in S$ such that $r_k < r + 1/k$. According to the definition of S, from $r_k \in S$ we conclude that if $|a - a'| \leq \delta$ and $|b - b'| \leq \delta$, then $|f(a,b) - f(a',b')| \leq r_k$. Letting $k \to \infty$, we conclude that $|f(a,b) - f(a',b')| \leq \lim_k r_k = r$. Q.E.D.

Proof of Theorem 1.
1°. Let us first prove that, as in the case of $t-$norm, we have $f(a,b) \leq \min(a,b)$ for any $\&-$operation f. Since $f(a,b) \leq a$ and $f(a,b) = f(b,a) \leq b$, it follows that $f(a,b) \leq \min(a,b)$.
2°. Next, $r_{\min}(\delta) = \delta$.

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Indeed, for $|a - a'| \le \delta$, we have $a \le a' + \delta$ and likewise $b \le b' + \delta$. Hence, $\min(a,b) \le \min(a' + \delta, b' + \delta) = \min(a', b') + \delta$, therefore, $\min(a,b) \le \min(a',b') + \delta$. Likewise, $\min(a',b') \le \min(a,b) + \delta$, so $-\delta \le \min(a,b) - \min(a',b') \le \delta$, and $|\min(a,b) - \min(a',b')| \le \delta$. Take $a = b = \delta$, $a' = b' = 0$. Then $|\min(a,b) - \min(a',b')| = \delta$, and therefore, the output error is precisely δ. So, we cannot take $\alpha < \delta$, and so the δ-robustness of min is really equal to δ.

$3°$. For every &-operation $f(a,b)$: $r_f(\delta) \ge r_{\min}(\delta) = \delta$.
Suppose that for some $\delta \in (0,1)$, $r_f(\delta) < \delta$. This means that if $|a - a'| \le \delta$ and $|b - b'| \le \delta$, then $|f(a,b) - f(a',b')| \le r_f(\delta) < \delta$. In particular, if we take $a = b = 1$ and $a' = b' = 1 - \delta$, we conclude that $|f(1,1) - f(1-\delta, 1-\delta)| < \delta$. But according to the definition of a &-operation, $f(1,1) = 1$, therefore, this inequality turns into $|1 - f(1-\delta,1-\delta)| < \delta$. Hence, $1 - f(1-\delta,1-\delta) \le |1 - f(1-\delta,1-\delta)| < \delta$, therefore, $f(1-\delta,1-\delta) > 1-\delta$. But we have already proved in $1°$ that $f(a,b) \le \min(a,b)$, therefore, $f(1-\delta,1-\delta) \le 1-\delta$. These two inequalities contradict to each other. Therefore, our assumption that $r_f(\delta) < \delta$ is incorrect. Hence, $r_f(\delta) \ge \delta$.

$4°$. $\min(a,b)$ is the only &-operation, for which $r_f(\delta) = \delta$ for all δ.
Indeed, suppose that f is different from min. Then for some a and b, $f(a,b) \ne \min(a,b)$, hence $f(a,b) < \min(a,b)$. Without loss of generality, assume $a \le b$, resulting in $f(a,b) < \min(a,b) = a$. For $a' = b' = 1$, we have $|a - a'| = 1 - a$, $|b - b'| = 1 - b \le 1 - a$, but $|f(a,b) - f(a',b')| = 1 - f(a,b) > 1 - a$. So, for $\delta_0 = 1 - a$, $r_f(\delta_0) > \delta_0$. Q.E.D.

Proof of Theorem 2 is similar.

Proof of Theorem 3. We have already proved (see part $4°$ of the proof of Theorem 1) that if an &-operation $f(a,b)$ is different from $\min(a,b)$, then $r_f(\delta_0) > \delta_0$ for some δ_0. So, in order to prove the Theorem it is sufficient to prove the following Lemma:

Lemma. *If $r_f(\delta_0) > \delta_0$ for some $\delta > 0$, then there exists a positive real number $\Delta > 0$ and positive real number $C < 1$ such that for all $\delta < \Delta$, $r_{\min}(\delta) \le C r_f(\delta)$.*

Proof of the Lemma. As we have already noticed, δ-robustness coincides with the modulus of continuity of f. The modulus of continuity is a subadditive function [13], i.e., $r_f(\delta_1+\delta_2+\ldots+\delta_n) \le r_f(\delta_1)+r_f(\delta_2)+\ldots+\delta_n$ for all $\delta_1, \ldots, \delta_n > 0$. In particular, for $\delta_1 = \delta_2 = \ldots = \delta_n = \delta_0/n$, we conclude that $r_f(\delta_0) \le n r_f(\delta_0/n)$. Therefore, $r_f(\delta_0/n) \ge r_f(\delta_0)/n$. If we denote $r_f(\delta_0)$ by D, then this inequality takes the form $r_f(\delta_0/n) \ge D/n$.

In order to continue the proof, we need to use one more property of the modulus of continuity [13]: if $\delta < \delta'$, then $r_f(\delta) \le r_f(\delta')$.
Let us now take any real number C between $c = \delta_0/D$ and 1 ($c < C < 1$), and prove that there exists a $\Delta > 0$ such that for all $\delta < \Delta$, we have $\delta \le C r_f(\delta)$ (or, equivalently, $r_f(\delta) \ge \delta/C$).
We already know how to estimate the values of $r_f(\delta)$ for $\delta = \delta_0/n$, where $n = 1,2,3,\ldots$ So, to get the estimates for arbitrary δ, we can use these known estimates. For every $\delta < \delta_0$, we want to find an n such that $\delta_0/(n+1) \le \delta \le \delta_0/n$. This inequality is equivalent to $(n+1)/\delta_0 \ge 1/\delta \ge n\delta_0$, which, after multiplying both sides by δ_0, turns out to be equivalent to the inequality $n \le \delta_0/\delta \le n+1$. Therefore, we can take for n the integer part $\lfloor \delta_0/\delta \rfloor$ of the ratio δ_0/δ. From monotonicity, we can conclude that $r_f(\delta) \ge r_f(\delta_0/(n+1))$. We have already proved that $r_f(\delta_0/(n+1)) \ge D/(n+1)$. Therefore, $r_f(\delta) \ge D/(n+1)$. We defined c as $c = \delta_0/D$; so, $D = \delta_0/c$, so $r_f(\delta) \ge \delta_0/(c(n+1))$.
We want to get an inequality $r_f(\delta) \ge \delta/C$. We will be able to deduce this inequality from the one that we have just proved if $\delta_0/(c(n+1)) \ge \delta/C$. Since $\delta \le \delta_0/n$, this inequality is valid if $\delta_0/Cn \le \delta_0/(c(n+1))$. Dividing both sides by δ_0 and then inverting both sides, we get an equivalent inequality $Cn \ge c(n+1)$, which, in its turn, is equivalent to $(C-c)n \ge c$ and $n \ge c/(C-c)$. Therefore, if $n \ge c/(C-c)$, then for $\delta \le \delta_0/n$ we get the desired inequality $r_f(\delta) \ge \delta/c$.
The inequality $n \ge c/(C-c)$ is valid for all n starting from $N = \lfloor c/(C-c) \rfloor + 1$. Therefore, the desired inequality $r_f(\delta) \ge \delta/c$ is true for all $\delta < \Delta$, where $\Delta = \delta_0/N$. Q.E.D.

Proof of Theorem 4 is similar.

Before proving Theorem 5 let us prove Proposition 2.

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Proof of Proposition 2.
1°. Let us first prove that if f and g are dual, i.e., $g(a,b) = 1-f(1-a,1-b)$, then for every δ, $r_g(\delta) \geq r_f(\delta)$. Indeed, suppose that $|a-a'| \leq \delta$ and $|b-b'| \leq \delta$, and let us prove that $|g(a,b)-g(a',b')| \leq r_f(\delta)$. Since $|a-a'| \leq \delta$ and $|b-b'| \leq \delta$, we have $|A-A'| = |a-a'| \leq \delta$ and $|B-B'| = |b-b'| \leq \delta$, where we denoted $A = 1-a$, $A' = 1-a'$, $B = 1-b$, and $B' = 1-b'$. Due to the definition of $r_f(\delta)$, we can conclude that $|f(A,B)-f(A',B')| \leq r_f(\delta)$. But $g(a,b) = 1-f(A,B)$ and $g(a',b') = 1-f(A',B')$, therefore $|g(a,b)-g(a',b')| = |f(A,B)-f(A',B')| \leq r_f(\delta)$. So, for $\alpha = r_f(\delta)$, if $|a-a'| \leq \delta$ and $|b-b'| \leq \delta$, then $|g(a,b)-g(a',b')| \leq \alpha$. Since $r_g(\delta)$ is defined as the smallest of all α with this property, we conclude that $r_g(\delta) \leq r_f(\delta)$.
2°. One can easily check that if g is dual to f, then f is dual to g. Therefore, we have both $r_g(\delta) \leq r_f(\delta)$ and $r_f(\delta) \leq r_g(\delta)$, hence $r_g(\delta) = r_g(\delta)$. Q.E.D.

Proof of Theorem 5.
1) We must prove, first, that if $|a-a'| \leq \delta$ and $|b-b'| \leq \delta$, then $|ab-a'b'| \leq 2\delta-\delta^2$, and, second, that there exist such a, b, a', b' for which $|a-a'| \leq \delta$, $|b-b'| \leq \delta$, and $|ab-a'b'| = 2\delta-\delta^2$.
The second statement is easy to prove: take $a = b = 1, a' = b' = 1-\delta$, then $|ab-a'b'| = 1-(1-\delta)^2 = 2\delta-\delta^2$. Let us now prove the first one.
Let us denote $|a-a'|$ by Δ_a, and $|b-b'|$ by Δ_b. Then $\Delta_a \leq \delta$ and $\Delta_b \leq \delta$. Without losing any generality we can assume that $a \geq a'$. Then $a' = a-\Delta_a$. With respect to b and b', there are two possible cases: $b \geq b'$ and $b < b'$. Let us consider both of them.
If $b \geq b'$, then $b' = b-\Delta_b$, and $ab \geq a'b'$, so the desired absolute value $d = |ab-a'b'|$ can be computed as follows: $d = |ab-a'b'| = ab-a'b' = ab-(a-\Delta_a)(b-\Delta_b) = a\Delta_b+b\Delta_a-\Delta_a\Delta_b$. Since $a \leq 1$ and $b \leq 1$, we have $d \leq \Delta_a+\Delta_b-\Delta_a\Delta_b$. The right-hand side of this inequality can be expressed as $1-(1-\Delta_a)(1-\Delta_b)$. Therefore, it is a monotonely increasing function of both Δ_a and Δ_b. So, its maximal value is attained when both of these variables take their biggest possible values. Since $\Delta_a \leq \delta$ and $\Delta_b \leq \delta$, the maximal possible value is attained when $\Delta_a = \Delta_b = \delta$, and is equal to $2\delta-\delta^2$. Therefore, $d \leq \Delta_a+\Delta_b-\Delta_a\Delta_b \leq 2\delta-\delta^2$. So for this case the desired inequality is proved.
Let us now consider the case when $b < b'$. Then $b = b'-\Delta_b$, and $d = |ab-a'b'| = |a(b'-\Delta_b)-(a-\Delta_a)b'| = |a\Delta_b-b'\Delta_a|$. Let us consider two subcases: when the expression under the absolute value is positive or negative, i.e., when $a\Delta_b \geq b'\Delta_a$ and $a\Delta_b < b'\Delta_a$. In the first subcase, $d = a\Delta_b-b'\Delta_a$, therefore, $d \leq b'\Delta_a$. Since $\Delta_a \leq \delta$ and $b' \leq 1$, we get $d \leq \delta$.
In the second subcase similarly $d = b'\Delta_a-a\Delta_b \leq b'\Delta_a \leq \delta$. So, in both cases $d \leq \delta$.
So, to complete the proof, it is sufficient to show that $\delta \leq 2\delta-\delta^2$ for all δ from 0 to 1. Indeed, by dividing both sides by δ and moving all terms to the right-hand side, we conclude that this inequality is equivalent to $0 \leq 1-\delta$, which is certainly true for $\delta \leq 1$.

2) follows from 1) and Proposition 2.

3) Let us first consider the case, when $\delta < 1/2$. Then $2\delta < 1$, and $\min(2\delta, 1) = 2\delta$. Let us prove that in this case, if $|a-a'| \leq \delta$ and $|b-b'| \leq \delta$, then $|f(a,b)-f(a',b')| \leq 2\delta$.
Indeed, if $|a-a'| \leq \delta$ and $|b-b'| \leq \delta$, then $|(a+b)-(a'+b')| = |((a-a')+(b-b')| \leq 2\delta$. In particular, this means that $a'+b' \leq a+b+2\delta$. Evidently, $a'+b' \leq 1$, therefore, $a'+b' \leq 1 < 1+2\delta$. So, $a'+b'$ is not bigger than the smallest of these two numbers: $a'+b' \leq \min(a+b+2\delta, 1+2\delta)$. But $\min(a+b+2\delta, 1+2\delta) = \min(a+b, 1)+2\delta = f(a,b)+2\delta$. So, $a'+b' \leq f(a,b)+2\delta$. Since $f(a',b') = \min(a'+b', 1)$ and therefore, $f(a',b') \leq a'+b'$, we conclude that $f(a',b') \leq f(a,b)+2\delta$. In a similar manner we can prove that $f(a,b) \leq f(a',b')+2\delta$. Combining these two inequalities, we conclude that $|f(a,b)-f(a',b')| \leq 2\delta$. So, for $\delta < 1/2$, $r_f(\delta) \leq 2\delta$.
Let us now show that $\alpha = 2\delta$ is the smallest value, for which $\delta-$input uncertainty leads to a $\leq \alpha-$output error, and thus, $r_f(\delta) = 2\delta$. Indeed, if we take $a = b = 0, a' = b' = \delta$, then $|a-a'| \leq \delta$, $|b-b'| \leq 2\delta$, and $|f(a,b)-f(a',b')| = |0-2\delta| = 2\delta$, so the values $\alpha < 2\delta$ do not work in this case. So, for $\delta < 1/2$, we proved that $r_f(\delta) = 2\delta$.
Now let us consider the case when $\delta \geq 1/2$. In this case, $\min(2\delta, 1) = 1$. If we take $a = b = 0, a' = b' = \delta$, then $f(a,b) = 0$, $f(a',b') = 1$, $|a-a'| \leq \delta$, $|b-b'| \leq 2\delta$, and $|f(a,b)-f(a',b')| = |0-1| = 1$. Therefore, nothing smaller than 1 can serve as α, hence $r_f(\delta) = 1$. Q.E.D.

Proof of Theorem 6. Let us first prove that for every negation operation, $r_f(\delta) \geq \delta$ for all δ. For a standard negation operation $f_0(x) = 1-x$, $f_0(\delta) = 1-\delta$. So, let us consider three possible cases: $f(\delta) < 1-\delta$, $f(\delta) < 1-\delta$, and $f(\delta) > 1-\delta$.

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In the first case, for $a = 0$, $a' = \delta$, we have $|a - a'| \le \delta$ and $|f(a) - f(a')| = |1 - f(\delta)| > \delta$. Therefore, $r_f(\delta) \ge |f(a) - f(a')| > \delta$, and $r_f(\delta) > \delta$.
In the second case, likewise $r_f(\delta) \ge \delta$.
In the third case, for $a = 1$ and $a' = f(\delta)$, we have $|a - a'| \le \delta$, but $|f(a) - f(a')| = |0 - \delta| = \delta$. Therefore, $r_f(\delta) \ge |f(a) - f(a')| \ge \delta$, and $r_f(\delta) \ge \delta$. So, in all three cases, we have $r_f(\delta) \ge \delta$ for all δ. For $f(x) = 1 - x$, $r_f(\delta) = \delta$. So, it is sufficient to prove that if a negation operation is not standard, i.e., $f(x) \ne 1 - x$ for some x, then $r_f(\delta) > \delta$ for some δ. The case when $f(x) < 1 - x$ was already considered above, and in this case, as we have already proved, $r_f(\delta) > \delta$ for $\delta = x$.
Suppose now that $f(x) > 1 - x$. Then $x > 1 - f(x)$. So, for $a = 1$ and $a' = f(x)$, we have $|a - a'| = 1 - f(x)$, and $|f(a) - f(a')| = |0 - x| = x$. Therefore, for $y = 1 - f(x)$, $r_f(y) \ge |f(a) - f(a')| = x > y$, and $r_f(y) > y$. Q.E.D.

Proof of Theorem 7 follows easily from the same Lemma as the proof of Theorem 3.

Proof of Theorem 8. Let us first prove that D_C is a continuous mapping. For that it is sufficient to prove that the mappings $\mu \to \int \mu(x)\,dx$ and $\mu \to \int x\mu(x)\,dx$ are continuous, then D_C will be continuous as a ratio of two continuous mappings. Indeed, if $|\mu(x) - \mu'(x)| \le \delta$ for all $x \in [a, b]$, then $-\delta \le \mu(x) - \mu'(x) \le \delta$, hence $-\delta(b-a) \le \int(\mu(x) - \mu'(x))\,dx = \int \mu(x)\,dx - \int \mu'(x)\,dx \le (b-a)\delta$. Therefore, $|\int \mu(x)\,dx - \int \mu'(x)\,dx| \le (b-a)\delta$, and we can easily prove continuity with $\delta = \varepsilon/(b-a)$. Likewise, $|\int x\mu(x)\,dx - \int x\mu'(x)\,dx| \le \delta(\int_a^b |x|\,dx)$, so this expression is also continuous.
Let us now prove that center-of-maximum is not continuous. Indeed, let us take a trapezoidal function $\mu(x)$ that is equal to 0 for $|x| > 2$, to 1 for $|x| \le 1$, to $2 - |x|$ for $1 \le |x| \le 2$. Then $m_- = -1, m_+ = 1$, hence $D_{COM}(\mu) = 0$. For every δ, we can define a new function $\mu_\delta(x) = \mu(x)(1 - (\delta/3)(1 - |x|))$. For this new function, the maximum (equal to 1) is attained in only one point $x = 1$, so $D_{COM}(\mu_\delta) = 1$.
Since we are considering only the values from -1 to 2, we have $|1 - |x| \le 3$, hence $(\delta/3)(1 - |x|) \le \delta$, and $|mu(x) - \mu'(x)| \le \delta$. So, for $\varepsilon = 1/2$, no matter how small $\delta > 0$ we take, we can always find a new function μ_δ such that $\rho(\mu, \mu_\delta) \le \delta$, but $D_{COM}(\mu_\delta) - D_{COM}(\mu) = 1 > 1/2$. So D_{COM} is not continuous. Q.E.D.

Before proving Proposition 3 and Theorem 9, let us first prove Theorem 10.

Proof of Theorem 10. For $f_0(x) = x/v$, one can easily compute that $r_{f_0}(\delta) = \delta/v$. We assumed that f is different from f_0, so $f(x) \ne f_0(x)$ for some x. For this x, either $f(x) < x/v$, or $f(x) > x/v$. In the first case, for $a = x$ and $a' = v$, we have $|a - a'| = v - x$ and $|f(a) - f(a')| = 1 - f(x) > 1 - x/v = (v - x)/v$, so $r_f(\delta) > \delta/v$ for $\delta = 1 - x$.
In the second case, for $a = 0$ and $a' = x$, we have $|a - a'| = x$ and $|f(a) - f(a')| > x/v$, so $r_f(\delta) > \delta/v$ for $\delta = x$. In both cases, $r_f(\delta) > \delta/v$ for some $\delta > 0$. Arguing like in Lemma, we can find the desired C and Δ. Q.E.D.

Proof of Theorem 9 directly follows from Theorem 10.

Proof of Proposition 3. If $f(x) \ne x/v$, then the existence of the desired $g(x)$ follows from Theorem 9: we can take $g(x) = x/v$. So in order to prove this Proposition, it is sufficient to prove it for $f(x) = x/v$, i.e., it is sufficient to find a normalization $g(x)$ such that $r_g(\delta) < \delta/v$ for some $\delta > 0$.
Let us define the following function $F(x)$: $F(x) = 100/3x$ for $x \le 0.01$, $F(x) = 1/3$ for $0.01 \le x \le 0.495$, $F(x) = 1/3 + 100/3 * (x - 0.495)$ for $0.495 \le x \le 0.505$, $F(x) = 2/3$ for $0.505 \le x \le 0.99$, and $F(x) = 2/3 + 100/3 * (x - 0.99)$ for $x > 0.99$. This is a continuous function from [0,1] to [0,1]. Let us prove that for $g(x) = F(x)/v$, $r_g(0.4) \le 1/(3v) < 0.4/v$. In other words, we want to prove that if $|a - a'| \le 0.4$, then $|g(a) - g(a')| \le 1/(3v) < 0.4/v$.
Without losing generality, we can assume that $a < a'$; then $a' \le 0.4 + a$, and the desired inequality takes the form $g(a') - g(a) \le 1/(3v) < 0.4/v$. Let us consider all possible locations of a.

- If $0 \le a \le 0.01$, then $a' \le 0.4 + a \le 0.4 + 0.01$, and, therefore $g(a') \le g(0.45) = (1/3)/v$. Hence, $g(a') - g(a) \le g(a') \le (1/3)/v$.
- If $0.01 \le a \le 0.495$, then $g(a) = (1/3)/v$, and $a' \le 0.4 + 0.495 = 0.895$, hence $g(a') \le g(0.895) = (2/3)/v$. Therefore, $g(a') - g(a) \le g(0.895) - g(a) = (1/3)/v$.
- If $0.495 \le a \le 0.505$, then $g(a) \ge g(0.495) = (1/3)/v$; here $a' \le 0.505 + 0.4$, hence $g(a') \le g(0.905) = (2/3)/v)$. Therefore, $g(a') - g(a) \le g(a') - g(0.495) = (1/3)/v$.
- For the cases $0.505 \le a \le 0.99$ and $a \ge 0.99$, the proofs are similar.

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So, in all the cases, $|g(a) - g(a')| \leq 1/(3v) < 0.4/v$. Q.E.D.

Proof of Theorems 11–14. For the piecewise-linear functions, $r_{\mu_0}(\delta) = k\delta$, where $k = 1/(x_+ - x_0)$ for functions of type 1, $k = 1/(x_+ - a_+)$ for functions of type 2, and $k = 1/(x_+ - x_-)$ for functions of types 3 and 4. The fact that these functions are asymptotically more robust than the others follows from Lemma, just like in the proof of Theorem 9.

CONCLUSIONS

As far as combining degrees of belief of experts is concerned, in situations where estimates can vary drastically, it is reasonable to use *robust* fuzzy logic connectives, which are the least sensitive to these variations, i.e., for which the resulting intervals of uncertainty are the smallest possible. We have proved that in this situation, the dual pair $\min(a, b)$, $\max(a, b)$ are the most robust operations. Results are also given for choosing the most robust negation operations, defuzzification procedures, and membership functions.

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